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Non-commutative quantum mechanics and the Aharonov–Casher effect

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Abstract. In this work a new method is developed to investigate the Aharonov–Casher effect in a non-commutative space. It is shown that the holonomy receives non-trivial kinematical corrections.

1 Introduction

In the last few years, theories in non-commutative space have been studied extensively (for a review see [1]). Non-commutative field theories are related to M-theory compactification [2], string theory in non-trivial backgrounds [3] and the quantum Hall effect [4]. Inclusion of non-commutativity in quantum field theory can be achieved in two different ways: via the Moyal $\star$-product on the space of ordinary functions, or defining the field theory on a coordinate operator space which is intrinsically non-commutative [1,5]. The equivalence between the two approaches has been nicely described in [6]. A simple insight in the role of non-commutativity in field theory can be obtained by studying the one particle sector, which prompted interest in the study of non-commutative quantum mechanics [7–14]. In these studies some attention was paid to the Aharonov–Bohm effect [15]. If the non-commutative effects are important at very high energies, then one could posit a decoupling theorem that produces the standard quantum field theory as an effective field theory and that does not leave the non-commutative effects. However the experience from atomic and molecular physics strongly suggests that the decoupling is never complete and that the high energy effects appear in the effective action as topological remnants. Along these lines, the Aharonov–Bohm effect has already been investigated in a non-commutative space [16]. In this work, we will develop a new method to obtain the corrections to the topological phase of the Aharonov–Casher effect, where we know that in a commutative space the line spectrum does not depend on the relativistic nature of the dipoles.

This article is organized as follows. In Sect. 2, we discuss the Aharonov–Casher effect on a commutative space. In Sect. 3, the Aharonov–Casher effect in a non-commutative space is studied and a generalized formula for holonomy is given.

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2 The Aharonov–Casher effect

In 1984 Aharonov and Casher (AC) [17] pointed out that the wave function of a neutral particle with non-zero magnetic moment μ develops a topological phase when traveling in a closed path which encircles an infinitely long filament carrying a uniform charge density. The AC phase has been measured experimentally [18]. This phenomenon is similar to the Aharonov–Bohm (AB) effect. The similarities and the differences of these two phenomena and possible classical interpretations of the AC effect have been discussed by several authors [19–21]. In [17], the topological phase of the AC effect was derived by considering a neutral particle with a non-zero magnetic dipole moment moving in an electric field produced by an infinitely long filament carrying a uniform charge density. If the particle travels over a closed path which includes the filament, a topological phase will result. This phase is given by

$$\phi_{\mathrm{AC}} = \oint (\boldsymbol{\mu} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{r}, \tag{1}$$

where $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$ is the magnetic dipole moment and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where σ_i $(i = 1, 2, 3)$ are the 2×2 Pauli matrices. It is possible to arrange that the particle moves in the x–y plane and travels over a closed path which includes an infinite filament along the z-axis. The electric field in the point $\mathbf{r} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the direction of the positive x- and y-axes, is given as

$$\mathbf{E} = \frac{\lambda}{2\pi(x^2 + y^2)}(x\hat{i} + y\hat{j}), \tag{2}$$

where λ is the linear charge density of the filament and the phase is given by

$$\phi_{\mathrm{AC}} = \mu\sigma_3 \oint (\hat{k} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{r} = \mu\sigma_3\lambda, \tag{3}$$

where $\hat{k}$ is a unit vector along the z-axis. This phase is purely quantum mechanical and has no classical interpretation. The appearance of σ_3 in the phase represents the spin degrees of freedom. We see that different components acquire phases with different signs. This is also one of the points that distinguishes the AC effect from the AB effect [22]. In this part, we briefly explain a method for obtaining (3). The equation of motion for a neutral spin half particle with a non-zero magnetic dipole moment moving in a static electric field $\mathbf{E}$ is given by

$$\left(\mathrm{i}\gamma_\mu \partial^\mu + \frac{1}{2}\mu\sigma_{\alpha\beta}F^{\alpha\beta} - m\right)\psi = 0, \tag{4}$$

or it can be written as

$$(\mathrm{i}\gamma_\mu \partial^\mu - \mathrm{i}\mu\gamma \cdot \mathbf{E}\gamma_0 - m)\psi = 0, \tag{5}$$

where $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ and the γ matrices are defined by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{6}$$

We define

$$\psi = \mathrm{e}^{\mathbf{a}f}\psi_0, \tag{7}$$

where $\mathbf{a}$ is the matrix to be determined below, f is a time independent scalar phase, and ψ_0 is a solution of the Dirac equation,

$$(\mathrm{i}\gamma_\mu \partial^\mu - m)\psi_0 = 0. \tag{8}$$

Writing ψ_0 in terms of ψ and multiplying (8) by $\mathrm{e}^{\mathbf{a}f}$ from the left, we obtain

$$\mathrm{e}^{\mathbf{a}f}(\mathrm{i}\gamma^\mu \partial_\mu - m)\mathrm{e}^{-\mathbf{a}f}\psi = 0, \tag{9}$$

$$(\mathrm{i}\mathrm{e}^{\mathbf{a}f}\gamma^\mu \mathrm{e}^{-\mathbf{a}f}\partial_\mu - \mathrm{i}\mathrm{e}^{\mathbf{a}f}\gamma^i \mathrm{e}^{-\mathbf{a}f}\mathbf{a}\,\partial_i f - m)\psi = 0. \tag{10}$$

Comparing (10) with (5), we find that $\mathbf{a}$ and f must satisfy

$$\mu\gamma \cdot \mathbf{E}\gamma_0 = (\gamma \cdot \nabla f)\mathbf{a} \quad , \quad \mathbf{a}\gamma_\mu = \gamma_\mu \mathbf{a}. \tag{11}$$

The matrix $\mathbf{a}$ can be expressed by some linear combination of the complete set of 4×4 matrices $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5$ and $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$. The second member of (11) cannot be satisfied if all γ_1, γ_2 and γ_3 are present in (10). However, it is possible to satisfy it if the problem in question can be reduced to the planar one. This indicates that the AC topological phase can arise only in two spatial dimensions. Therefore, let us consider a particle moving in the x–y plane in which case only the matrices γ_1 and γ_2 are present in (11), and moreover, $\partial_3\psi$ and E_3 vanish. The choice $-\mathrm{i}\sigma_{12}\gamma_0$ represents a consistent Ansatz. From the first equation in (11), we get

$$\nabla f = \mu(\hat{k} \times \mathbf{E}), \tag{12}$$

and the phase is given by

$$\begin{aligned} \phi^{(0)} &= \sigma_{12}\gamma_0 \oint \nabla f \cdot \mathrm{d}\mathbf{r} \\ &= \mu\sigma_{12}\gamma_0 \oint (\hat{k} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{r} \\ &= \mu \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \oint (\hat{k} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{r}. \end{aligned} \tag{13}$$

3 The Aharonov–Casher effect in a non-commutative space

The non-commutative Moyal spaces can be realized as spaces where the coordinate operator $\hat{x}^\mu$ satisfies the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{14}$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor of space dimension $(\text{length})^2$. We note that space-time non-commutativity, $\theta^{0i} \neq 0$, may lead to some problems with unitarity and causality. Such problems do not occur for the quantum mechanics on a non-commutative space with a usual commutative time coordinate. The non-commutative models specified by (14) can be realized in terms of a $\star$-product: the commutative algebra of functions with the usual product $f(x)g(x)$ is replaced by the $\star$-product Moyal algebra:

$$(f \star g)(x) = \exp\left[\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right] f(x)g(y)|_{x=y}. \tag{15}$$

As for the phase space, inferred from string theory, we choose

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{16}$$

The non-commutative quantum mechanics can be defined by [7–14]

$$H(p, x) \star \psi(x) = E\psi(x). \tag{17}$$

The equation of motion for a neutral spin half particle with a non-zero magnetic dipole moment moving in a static electric field $\mathbf{E}$ is given by

$$\left(\mathrm{i}\gamma_\mu \partial^\mu + \frac{1}{2}\mu\sigma_{\alpha\beta}F^{\alpha\beta} - m\right) \star \psi = 0, \tag{18}$$

or it can be written as

$$(\mathrm{i}\gamma_\mu \partial^\mu - \mathrm{i}\mu\gamma \cdot \mathbf{E}\gamma_0 - m) \star \psi = 0. \tag{19}$$

We define

$$\psi = \mathrm{e}^{\mathbf{a}f}\psi_0, \tag{20}$$

where $\mathbf{a}$ is the matrix already defined ($\mathbf{a}\gamma_\mu = \gamma_\mu\mathbf{a}$), f is a time independent scalar phase, and ψ_0 is a solution of the Dirac equation

$$(\mathrm{i}\gamma_\mu \partial^\mu - m)\psi_0 = 0, \tag{21}$$

and (19) can be written as

$$(\mathrm{i}\gamma_\mu \partial^\mu \mathrm{e}^{\mathbf{a}f})\psi_0 - (\mathrm{i}\mu\gamma \cdot \mathbf{E}\gamma_0) \star (\,\mathrm{e}^{\mathbf{a}f}\psi_0) = 0. \tag{22}$$

After expanding the second term in (22) up to the first order of the non-commutativity parameter $\theta_{ij} = \theta\epsilon_{ij}$ and defining k_j as

$$\partial_j\psi_0 = (\mathrm{i}k_j)\psi_0, \tag{23}$$

the final result up to first order in θ is given by

$$\Big[\mathrm{i}(\gamma^i\partial_i\mathbf{a}f) - \mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0 - \frac{\mathrm{i}}{2}\theta^{ij}[\partial_i(\mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0)](\mathrm{i}k_j) - \frac{\mathrm{i}}{2}\theta^{ij}[\partial_i(\mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0)]\mathbf{a}\,\partial_j f\Big]\ \mathrm{e}^{\mathbf{a}f}\psi_0 = 0, \tag{24}$$

or we get the following equation ($\mathbf{a}\gamma_\mu = \gamma_\mu\mathbf{a}$):

$$\Big[\mathrm{i}(\gamma^i\partial_i\mathbf{a}f) - \mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0 - \frac{\mathrm{i}}{2}\theta^{ij}[\partial_i(\mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0)](\mathrm{i}k_j) - \frac{\mathrm{i}}{2}\theta^{ij}[\partial_i(\mathrm{i}\mu\gamma\cdot\mathbf{E}\gamma_0)]\mathbf{a}\,\partial_j f\Big]\psi_0 = 0. \tag{25}$$

It should be noted that expansion of $\mathbf{E}$ up to first order in θ leads to an additive correction to the commutative holonomy and does not cause a new non-topological behavior. A similar situation arises in the non-commutative Aharonov–Bohm effect. By expanding f up to first order in θ,

$$f = f^{(0)} + \theta f^{(1)} + ..., \tag{26}$$

we obtain the following equations:

$$[\mu\gamma\cdot\mathbf{E}\gamma_0 - (\gamma^i\ \partial_i f^{(0)})\ \mathbf{a}]\psi_0 = 0, \tag{27}$$

which is equivalent to (11) and

$$\Big[\gamma^i\mathbf{a}\ \partial_i f^{(1)} + \frac{1}{2}\mu\varepsilon^{ij}k_j\partial_i(\mu\gamma\cdot\mathbf{E}\gamma_0) - \frac{\mathrm{i}}{2}\varepsilon^{ij}\partial_i(\mu\gamma\cdot\mathbf{E}\gamma_0)\mathbf{a}\ \partial_j f^{(0)}\Big]\psi_0 = 0. \tag{28}$$

By choosing $\mathbf{a} = -\mathrm{i}\sigma_{12}\gamma_0$ and after a straightforward calculation we get

$$\nabla f^{(0)} = \mu(\hat{k}\times\mathbf{E}), \tag{29}$$

and the phase is given by

$$\begin{aligned}\phi^{(0)} &= \sigma_{12}\gamma_0\oint\nabla f^{(0)}\cdot\mathrm{d}\mathbf{r}\\ &= \mu\sigma_{12}\gamma_0\oint(\hat{k}\times\mathbf{E})\cdot\mathrm{d}\mathbf{r}\\ &= \mu\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}\oint(\hat{k}\times\mathbf{E})\cdot\mathrm{d}\mathbf{r}.\end{aligned} \tag{30}$$

Substituting (29) in (28) yields

$$\Big[\mathrm{i}\gamma^i\sigma_{12}\gamma_0\partial_i f^{(1)} - \frac{1}{2}\varepsilon^{ij}k_j\partial_i(\mu\gamma\cdot\mathbf{E}\gamma_0) + \frac{1}{2}\varepsilon^{ij}\partial_i(\mu\gamma\cdot\mathbf{E}\gamma_0)\sigma_{12}\gamma_0\mu(\hat{k}\times\mathbf{E})_j\Big]\psi_0 = 0. \tag{31}$$

After a long but straightforward calculation, the following correction to $\phi^{(0)}$ for a neutral particle with non-zero magnetic dipole moment μ and with spin up or down ($\mp$) is obtained:

$$\begin{aligned}\triangle\phi_\theta &= \theta\sigma_{12}\gamma_0\oint\nabla f^{(1)}\cdot\mathrm{d}\mathbf{r}\\ &= \frac{\theta}{2}\sigma_{12}\gamma_0\varepsilon^{ij}\ \Big(\mu\oint k_j(\partial_i E_2\mathrm{d}x_1 - \partial_i E_1\mathrm{d}x_2)\\ &\quad\mp\oint[(\mu\partial_i E_2)\ \mu(\hat{k}\times\mathbf{E})_j\mathrm{d}x_1\\ &\quad-(\mu\partial_i E_1)\ \mu(\hat{k}\times\mathbf{E})_j\mathrm{d}x_2]\Big).\end{aligned} \tag{32}$$

The first term is a velocity dependent correction and does not have the topological properties of the commutative AC effect and could modify the phase shift. The second term is a correction to the vortex and does not contribute to the line spectrum. Using the following notation:

$$\mathbf{k}\propto\mathbf{v},\quad \mu(\hat{k}\times\mathbf{E})\propto\mathbf{A}^{(0)}, \tag{33}$$

the integral in (32) can be mapped to the corrections which have already been obtained for the Aharonov–Bohm effect in [16], (2.19), where $\mathbf{v}$ and $\mathbf{A}^{(0)}$ are the velocity of the particle and the vector potential in the Aharonov–Bohm effect.

The total phase shift for the AC effect is given by

$$\phi_{\mathrm{total}} = \phi_0 + \triangle\phi_\theta, \tag{34}$$

$$\phi_0 = \mu\lambda, \tag{35}$$

where λ is the charge per unit length along the filament. $\triangle\phi_\theta$ can be estimated for a circular path and the contribution to the shift coming from non-commutativity, relative to the usual shift of phase, is given by

$$\frac{\triangle\phi_\theta}{\phi_0}\simeq\frac{\theta}{R\lambda_{\mathrm{n}}} + \frac{\theta\phi_0}{\pi R^2}, \tag{36}$$

where λ_{n} is the wavelength of the neutral particle and R is the radius of the approximate path. The non-commutative contributions are very tiny. The experimental observations on the AC phase shift [18] can be used to put a limit on the non-commutativity parameter θ. In [18], a crystal neutron interferometer has been used and thermal neutrons travel in half of their paths in a constant electric field. The path is not circular but can be approximated by a circle with a radius which is about 1 cm ($L = 2.53$ cm; $R\simeq L\sin(22.5°)$ in [18]). Fitting (36) into the accuracy bound of the experiment [18], we obtain

$$\frac{\triangle\phi_\theta}{\phi_0}\leq 25\%, \tag{37}$$

$$\sqrt{\theta}\leq 10^7\ \mathrm{GeV}^{-1}. \tag{38}$$

The low energy (thermal) neutrons in the experimental test of the AC effect [18] cause a higher limit for θ as compared to other limits recently obtained [23–25]; however, we hope that future experiments with high energy neutrons lead to a better limit for the non-commutativity parameter in the AC effect.

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